Chiral QED out of matter

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Abstract

In a previous paper we have shown how the Wilsonian renormalization group naturally leads to the equivalence of the standard QED with a matter-only theory. In this paper we give an improved explanation of the equivalence and discuss, as an example, the equivalence of a chiral QED in the Higgs phase with a matter-only theory. Ignoring the contributions suppressed by the negative powers of a UV cutoff, the matter-only theory is equivalent to the perturbatively renormalizable chiral QED with two complex Higgs fields. In the matter-only theory chiral anomaly arises without elementary gauge fields.

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1   Introduction

The possibility of writing down a matter-only theory which contains all the interactions of nature was first considered by Heisenberg in his study of a unified field theory.[1] Along a somewhat different line of thought, motivated by the model of dynamical symmetry breaking by Nambu and Jona-Lasinio[2], Bjorken proposed a fermionic theory without an elementary gauge field which might explain the masslessness of the photon as a consequence of a symmetry breaking.[3] He actually found his model equivalent to the standard QED, and the equivalence was further discussed by Bialynicki-Birula, Lurié and Macfarlane, and Guralnik among others.[4] Similarly, the equivalence of the Nambu-Jona-Lasinio model with a manifestly renormalizable Yukawa theory was shown by Eguchi.[5]

A clear explanation of such equivalence between a manifestly renormalizable theory and an apparently non-renormalizable theory was only relatively recently given in the work of Hasenfratz et al.[6] from the viewpoint of the Wilsonian renormalization group.[7] It is this viewpoint which was extended to the equivalence between QED and the Bjorken model in the previous paper by the author.[8]

The purpose of the present paper is two-fold. First we improve the explanation of the equivalence between a manifestly renormalizable model and an apparently non-renormalizable model. Using the $\frac{1}{N}$ expansions to leading order, we give a simple yet well defined procedure for building a non-renormalizable model which is equivalent to the original renormalizable model. Though the equivalence is guaranteed only to leading order in $\frac{1}{N}$, the possibility of extending the equivalence beyond the leading order is supported by the general renormalization group argument. Second, as a concrete example, we show how to use the above procedure to construct a matter-only lagrangian equivalent to a chiral QED which has a much richer structure than the models of QED discussed in the previous paper.

The paper is organized as follows. In sect. 2 we give a simpler explanation of the equivalence of the Bjorken model with QED than was given in the previous paper. Using this example, we construct a simple procedure for constructing a matter-only theory equivalent to the original model to leading order in $\frac{1}{N}$. In sect. 3 we summarize the relevant properties of a chiral QED with two flavors of chiral fermions and two complex scalar fields. In sect. 4 we follow the procedure given in sect. 2 to construct a matter-only theory equivalent to the chiral QED of sect. 3. We discuss the realization of chiral anomaly in the matter-only model in sect. 5 before we conclude the paper in sect. 6. Three appendices are given for completeness.

We work in the four dimensional euclidean space throughout and use the same convention for the spinors as in ref. [8].

2 Equivalence revisited

The equivalence of QED with a matter-only theory was discussed from the Wilsonian RG (renormalization group) viewpoint in the previous paper [8]. In this section we wish to give an improved explanation of the equivalence.

The standard QED, which is manifestly renormalizable by perturbation theory, is defined by the lagrangian

$$
\begin{aligned}
\mathcal{L}_{QED} \;=\;& \frac{1}{4e_0^2}F_{\mu\nu}^2 + \frac{1}{2e_0^2\xi_0}(\partial_\mu A_\mu)^2 + \frac{m_0^2}{2e_0^2}A_\mu^2 + \frac{\lambda}{N(4\pi)^2}\frac{(A_\mu^2)^2}{8} \\
&+ \overline{\psi}^I\left(\frac{1}{i}\slashed{\partial} + \frac{1}{\sqrt{N}}\slashed{A} + iM\right)\psi^I
\end{aligned}
\tag{1}
$$

where $I = 1, ..., N$. We regularize the theory with a momentum cutoff Λ. To leading order in $\frac{1}{N}$ the theory is renormalized by

$$
\frac{(4\pi)^2}{e^2} \;\equiv\; \frac{4}{3}\ln\frac{\Lambda_0^2}{\mu^2} = \frac{(4\pi)^2}{e_0^2} + \frac{4}{3}\ln\frac{\Lambda^2}{\mu^2} - 1
\tag{2}
$$

$$
\frac{m_\gamma^2}{e^2} \;=\; \frac{m_0^2}{e_0^2} - \frac{2}{(4\pi)^2}(\Lambda^2 - M^2)
\tag{3}
$$

$$
\frac{(4\pi)^2}{e^2\xi} \;=\; \frac{(4\pi)^2}{e_0^2\xi_0} + \frac{1}{3}
\tag{4}
$$

where m_γ is a finite photon mass, and μ is an arbitrary low energy renormalization scale. To satisfy the Ward identity we must choose

$$
\lambda = -\frac{4}{3}
\tag{5}
$$

to leading order in $\frac{1}{N}$ so that the four-point proper vertex of the photon field vanishes at zero external momenta. The momentum cutoff does not allow the shift of loop momenta, and it does not respect the gauge invariance of the lagrangian. Hence, we need to introduce the self-coupling λ to enforce the Ward identity.

Now, let us try to construct a matter-only model which is equivalent to the above, to leading order in $\frac{1}{N}$. There is no unique choice for the lagrangian, but one straightforward choice is the following:

$$
\mathcal{L}_{Bj} = \overline{\psi}^I\left(\frac{1}{i}\slashed{\partial} + iM\right)\psi^I - \frac{1}{2Nv^2}\left(\overline{\psi}^I\gamma_\mu\psi^I\right)^2 + \Delta\mathcal{L}
\tag{6}
$$

where

$$v^2 \equiv \frac{m_0^2}{e_0^2} = \frac{2}{(4\pi)^2}(\Lambda^2 - M^2) + \frac{m_\gamma^2}{e^2} \tag{7}$$

Here, the counterterms $\Delta\mathcal{L}$ are given by

$$\Delta\mathcal{L} = \frac{1}{4e_0^2}(\partial_\mu B_\nu - \partial_\mu B_\mu)^2 + \frac{1}{2e_0^2\xi_0}(\partial_\mu B_\mu)^2 + \frac{\lambda}{N(4\pi)^2}\frac{(B_\mu^2)^2}{8} \tag{8}$$

where

$$B_\mu \equiv -\frac{(4\pi)^2}{2\Lambda^2}\frac{1}{\sqrt{N}}\overline{\psi}^I\gamma_\mu\psi^I \tag{9}$$

We can understand the equivalence between $\mathcal{L}_{QED}$ and $\mathcal{L}_{Bj}$ from the viewpoint of the Wilsonian RG.[7] We notice that the lagrangian (6) has two distinct critical points. One is trivial: $M = 0$ and v arbitrary (except for v_c given below). The other parameters e_0^2, ξ_0, λ are also arbitrary. This critical point, which describes N free massless fermions, has codimension 1 in theory space, and the fermion mass is the sole relevant parameter. There is another non-trivial critical point, however, which is given by $M = 0$ and $v^2 = v_c^2$ (where $v_c^2 = \frac{2\Lambda^2}{(4\pi)^2}$ to leading order in $\frac{1}{N}$). The remaining parameters are arbitrary. This critical point corresponds to N free massless fermions and one free massless vector. The criticality has codimension 2 in theory space, and the two relevant parameters are the fermion and photon masses. In finding the non-trivial critical point of the matter-only theory, the $\frac{1}{N}$ expansions are helpful.[9] As long as we use perturbation expansions in powers of the four-fermi interaction, we cannot detect the second critical point. The Wilsonian RG tells us that arbitrary theories which are almost critical are completely characterized by the relevant and marginally irrelevant parameters. The corrections are suppressed by the negative powers of the cutoff. The critical point of $\mathcal{L}_{QED}$ at $M = 0$, $\frac{m_0^2}{e_0^2} = v_c^2$ and that of $\mathcal{L}_{Bj}$ at $M = 0$, $v^2 = v_c^2$ describe the same criticality; both run to the same fixed point under the RG. Therefore, the two lagrangians must define the same theory.[7]

The form of the matter-only lagrangian (6) is by no means unique. Any lagrangian in the neighborhood of the non-trivial critical point will do as long as it has enough degrees of freedom to allow for two marginally irrelevant parameters e_0^2, λ. In ref. [8] we have given a different but equivalent lagrangian.

The procedure to get $\mathcal{L}_{Bj}$ out of $\mathcal{L}_{QED}$ can be made systematic. We first extract the gaussian part of $\mathcal{L}_{QED}$ quadratic in the gauge field with no derivatives:

$$\mathcal{L}_{gauss} = \frac{v^2}{2}A_\mu^2 + A_\mu\frac{1}{\sqrt{N}}\overline{\psi}^I\gamma_\mu\psi^I \tag{10}$$

where $v^2 \equiv \frac{m_0^2}{e_0^2}$. The equation of motion for this lagrangian gives

$$A_\mu = B'_\mu \equiv -\frac{1}{\sqrt{N}v^2}\overline{\psi}^I \gamma_\mu \psi^I \qquad (11)$$

Now we can construct a matter-only lagrangian by substituting the above interpolating field B'_μ into A_μ in the remaining part of the lagrangian $\mathcal{L}_{QED}$:

$$\mathcal{L}_{matter} = \overline{\psi}^I \left(\frac{1}{i}\slashed{\partial} + iM\right)\psi^I + \frac{v^2}{2}\left(A_\mu - B'_\mu\right)^2 - \frac{v^2}{2}B_\mu^{'\,2}$$

$$+\frac{1}{4e_0^2}(\partial_\mu B'_\nu - \partial_\mu B'_\mu)^2 + \frac{1}{2e_0^2\xi_0}(\partial_\mu B'_\mu)^2 + \frac{\lambda}{N(4\pi)^2}\frac{(B_\mu^{'\,2})^2}{8} \qquad (12)$$

Note that the above lagrangian $\mathcal{L}_{matter}$ is quadratic in A_μ which plays the role of an auxiliary field. The equation of motion for A_μ is given by Eq. (11), and to leading order in $\frac{1}{N}$ we can substitute the equation of motion inside the correlation functions. This is because the proper two-point function of A_μ and B'_ν is given by the diagram in Fig. 1 to leading order in $\frac{1}{N}$, and the proper part of the correlation function of $\frac{1}{n!}\left(B_\mu^{'\,2}/2\right)^n$ with $2n$ A_μ fields is given by 1 (except for the tensorial factor) to leading order in $\frac{1}{N}$. (Fig. 2) The errors of this substitution are suppressed by negative powers of the cutoff. Integrating over the auxiliary field A_μ, we obtain the lagrangian $\mathcal{L}_{Bj}$ (6). Except for the mass term B_μ^2, we can replace B'_μ by B_μ since the two differ only by a normalization which is unity up to $\frac{\mu^2}{\Lambda^2}$. Thus, to leading order in $\frac{1}{N}$, the two lagrangians $\mathcal{L}_{QED}$ (1) and $\mathcal{L}_{Bj}$ (6) are equivalent.
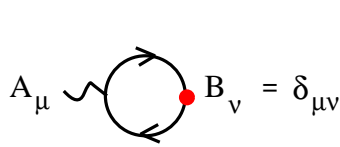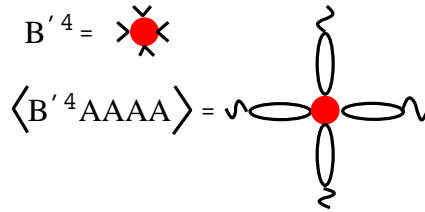

Fig. 1 Proper two-point vertex

Fig. 2 Proper four-point vertex of the fourth power of B'

While the equivalence between (1) and (6) is strictly valid only to leading order in $\frac{1}{N}$, the equivalence can be generalized beyond the leading order by choosing appropriate values for the parameters of (6), as is assured by the general Wilsonian RG argument.

Finally we note that the lagrangian of the matter-only theory simplifies somewhat if we raise the momentum cutoff Λ to the Landau scale Λ_0 defined by (2). Then the bare charge e_0 diverges, and the lagrangian (6) becomes

$$\mathcal{L}'_{Bj} = \overline{\psi}^I \left(\frac{1}{i} \slashed{\partial} + iM \right) \psi^I - \frac{1}{2Nv^2} \left(\overline{\psi}^I \gamma_\mu \psi^I \right)^2 + \frac{\lambda}{N(4\pi)^2} \frac{(B_\mu^2)^2}{8} \qquad (13)$$

This choice of the cutoff is known as the "compositeness condition" in the literature.[4]

We hope that the reader is convinced that any gauge theory (at least as long as it is abelian) can be rewritten as a matter-only theory. For completeness we also derive the equivalence between the manifestly renormalizable Yukawa model with the Nambu-Jona-Lasinio model along the above line in Appendix A.

3 Chiral QED with a momentum cutoff

We consider a QED with chiral fermions in the Higgs phase.[1] Contrary to the examples considered in ref. [8] which had an explicit photon mass, here the photon mass results dynamically from the Higgs mechanism.

For our purposes it is important to use a momentum cutoff regularization. As we have seen in the previous section, composite interpolating fields such as B_μ (9) play an important role in constructing matter-only theories, and their definitions need a cutoff explicitly. With a momentum cutoff Λ, the theory is defined by the following lagrangian

$$\mathcal{L}_{chiral} = \frac{1}{4e_0^2} F_{\mu\nu}^2 + \frac{1}{2e_0^2\xi_0}(\partial_\mu A_\mu)^2 + \frac{m_0^2}{2e_0^2}A_\mu^2$$

$$+\overline{u}^I \left(\frac{1}{i}\slashed{\partial} + \frac{1}{\sqrt{N}}\slashed{A}\frac{1+\gamma_5}{2} \right) u^I + \overline{d}^I \left(\frac{1}{i}\slashed{\partial} - \frac{1}{\sqrt{N}}\slashed{A}\frac{1+\gamma_5}{2} \right) d^I$$

$$+\frac{1}{g_{u,0}^2} \left(\partial_\mu - \frac{i}{\sqrt{N}}A_\mu \right) \phi_u^* \cdot \left(\partial_\mu + \frac{i}{\sqrt{N}}A_\mu \right) \phi_u$$

$$+\frac{1}{g_{d,0}^2} \left(\partial_\mu + \frac{i}{\sqrt{N}}A_\mu \right) \phi_d^* \cdot \left(\partial_\mu - \frac{i}{\sqrt{N}}A_\mu \right) \phi_d$$

$$+\frac{i}{\sqrt{N}} \left(\phi_u \overline{u}_R^I u_L^I + \phi_u^* \overline{u}_L^I u_R^I \right) + \frac{i}{\sqrt{N}} \left(\phi_d \overline{d}_R^I d_L^I + \phi_d^* \overline{d}_L^I d_R^I \right) \qquad (14)$$

$$+\frac{1}{N} \frac{\lambda_{u,0}}{4g_{u,0}^4} \left(\phi_u^* \phi_u \right)^2 + \frac{M_{u,0}^2}{g_{u,0}^2} \phi_u^* \phi_u + \frac{1}{N} \frac{\lambda_{d,0}}{4g_{d,0}^4} \left(\phi_d^* \phi_d \right)^2 + \frac{M_{d,0}^2}{g_{d,0}^2} \phi_d^* \phi_d$$

---

[1]We do not know of any appropriate reference or textbook to point to.

$$+\frac{1}{N}\frac{\tilde{\lambda}}{g_{u,0}^2 g_{d,0}^2}\left(\phi_u^*\phi_u\right)\left(\phi_d^*\phi_d\right)+\Delta\mathcal{L}$$

where $\Delta\mathcal{L}$ denotes the counterterms to be given below to compensate for the gauge non-invariance of the momentum cutoff Λ.

The lagrangian is invariant under the following global chiral transformations:

$$u_R^I \rightarrow e^{i\theta_u}u_R^I, \quad \phi_u \rightarrow e^{i\theta_u}\phi_u \tag{15}$$

$$d_R^I \rightarrow e^{i\theta_d}d_R^I, \quad \phi_d \rightarrow e^{i\theta_d}\phi_d \tag{16}$$

The off-diagonal part $\theta_u = -\theta_d$ can be gauged thanks to the anomaly cancellation between the two flavors u, d. We note that the above global invariance properties are preserved by the momentum cutoff regularization.

Though a non-vanishing $\tilde{\lambda}$ is allowed by the above chiral symmetry, we take $\tilde{\lambda} = 0$ and ignore the mixing between ϕ_u and ϕ_d in the rest of the paper. This is solely to simplify the calculations.

We now consider the Higgs phase in which both u and d acquire a mass. Denoting the fermion mass by m_i for $i = u, d$, we obtain the following relation to leading order in $\frac{1}{N}$:

$$\frac{1}{2}\lambda_{i,0}\frac{m_i^2}{g_{i,0}^2} + M_{u,0}^2 = \frac{2}{(4\pi)^2}g_{i,0}^2\left(\Lambda^2 - m_i^2\ln\frac{\Lambda^2}{m_i^2}\right) \tag{17}$$

Thus, for the theory to be in the Higgs phase we must choose

$$M_{i,0}^2 < \frac{2}{(4\pi)^2}g_{i,0}^2\Lambda^2 \tag{18}$$

With the following shifts of fields the fields ρ_i, φ_i have vanishing expectation values:

$$\phi_i = \sqrt{N}m_i + \frac{g_{i,0}}{\sqrt{2}}\left(\rho_i + i\varphi_i\right) \tag{19}$$

To compensate for the non-gauge invariance of the momentum-cutoff regularization we introduce the following counterterms $\Delta\mathcal{L}$:

$$\Delta\mathcal{L} = C\frac{1}{8(4\pi)^2 N}\left(A_\mu^2\right)^2 + \frac{1}{(4\pi)^2 N}A_\mu^2\left(a_u\phi_u^*\phi_u + a_d\phi_d^*\phi_d\right) \tag{20}$$

To satisfy the Ward identities (which are summarized in Appendix C), we must make the following choice to leading order in $\frac{1}{N}$:

$$\frac{m_0^2}{e_0^2} = \frac{2\Lambda^2}{(4\pi)^2} \tag{21}$$

$$C = -\frac{4}{3}, \quad a_u = a_d = -\frac{1}{2} \tag{22}$$

Unlike the case of QED which has the photon mass as a free parameter, the mass m_0^2 is uniquely determined by the Ward identity.

To leading order in $\frac{1}{N}$ the renormalization is done as follows:

$$e = \sqrt{Z_3}\, e_0 \tag{23}$$

$$g_i = \sqrt{Z_i}\, g_{i,0} \quad (i = u, d) \tag{24}$$

$$\frac{\lambda_i}{4g_i^4} = \frac{\lambda_{i,0}}{4g_{i,0}^4} + \frac{1}{(4\pi)^2}\ln\frac{\Lambda^2}{\mu^2} \tag{25}$$

where the renormalization constants are defined by

$$Z_3 \equiv 1 - \frac{4}{3}\frac{e^2}{(4\pi)^2}\ln\frac{\Lambda^2}{\mu^2} = \frac{1}{1 + \frac{4}{3}\frac{e_0^2}{(4\pi)^2}\ln\frac{\Lambda^2}{\mu^2}} \tag{26}$$

$$Z_i \equiv 1 - \frac{g_i^2}{(4\pi)^2}\ln\frac{\Lambda^2}{\mu^2} = \frac{1}{1 + \frac{g_{i,0}^2}{(4\pi)^2}\ln\frac{\Lambda^2}{\mu^2}} \tag{27}$$

The fermion masses $m_i (i = u, d)$ are unrenormalized to leading order in $\frac{1}{N}$. The products $e_0 A_\mu$, $g_{i,0}\rho_i$, $g_{i,0}\varphi_i$ are also left unrenormalized. We summarize the results for the renormalized correlation functions in Appendix B.

4 Matter-only model

The lagrangian of the matter-only model can be obtained following the procedure given in sect. 2. The gaussian part of the lagrangian is given by

$$\mathcal{L}_{gauss} = \frac{v^2}{2}\left(A_\mu^2 - 2A_\mu B_\mu'\right)$$
$$+ V_u^2\left(\phi_u^*\phi_u - \Phi_u'^*\phi_u - \phi_u^*\Phi_u'\right) + V_d^2\left(\phi_d^*\phi_d - \Phi_d'^*\phi_d - \phi_d^*\Phi_d'\right) \tag{28}$$

where

$$v^2 \equiv \frac{m_0^2}{e_0^2} = \frac{2}{(4\pi)^2}\Lambda^2 \tag{29}$$

$$V_u^2 \equiv \frac{M_{u,0}^2}{g_{u,0}^2} = \frac{2}{(4\pi)^2}\Lambda^2 - 2m_u^2\left(\frac{\lambda_{u,0}}{4g_{u,0}^4} + \frac{1}{(4\pi)^2}\ln\frac{\Lambda^2}{m_u^2}\right) \tag{30}$$

$$V_d^2 \equiv \frac{M_{d,0}^2}{g_{d,0}^2} = \frac{2}{(4\pi)^2}\Lambda^2 - 2m_d^2\left(\frac{\lambda_{d,0}}{4g_{d,0}^4} + \frac{1}{(4\pi)^2}\ln\frac{\Lambda^2}{m_d^2}\right) \tag{31}$$

and

$$B'_\mu \equiv -\frac{1}{v^2}\frac{1}{\sqrt{N}}\left(\overline{u}_R^I\gamma_\mu u_R^I - \overline{d}_R^I\gamma_\mu d_R^I\right)$$

$$\Phi'_u \equiv -\frac{1}{V_u^2}\frac{1}{\sqrt{N}}i\overline{u}_L^I u_R^I \quad \Phi'_d \equiv -\frac{1}{V_d^2}\frac{1}{\sqrt{N}}i\overline{d}_L^I d_R^I \qquad (32)$$

Hence, we obtain the following matter-only lagrangian:

$$\mathcal{L}_{matter} = \overline{u}^I\frac{1}{i}\slashed{\partial}u^I + \overline{d}^I\frac{1}{i}\slashed{\partial}d^I - \frac{1}{2Nv^2}\left(\overline{u}_R^I\gamma_\mu u_R^I - \overline{d}_R^I\gamma_\mu d_R\right)^2$$

$$+\frac{1}{NV_u^2}\left(\overline{u}_L^I u_R^I\right)\cdot\left(\overline{u}_R^J u_L^J\right) + \frac{1}{NV_d^2}\left(\overline{d}_L^I d_R^I\right)\cdot\left(\overline{d}_R^J d_L^J\right) + \Delta\mathcal{L} \quad (33)$$

The counterterms are given by

$$\Delta\mathcal{L} = \frac{1}{e_0^2}\frac{1}{4}\left(\partial_\mu B_\nu - \partial_\nu B_\mu\right)^2 + \frac{1}{\xi_0 e_0^2}\frac{1}{2}(\partial_\mu B_\mu)^2 + \frac{C}{8(4\pi)^2 N}\left(B_\mu^2\right)^2$$

$$+\frac{1}{g_{u,0}^2}\left(\partial_\mu - i\frac{1}{\sqrt{N}}B_\mu\right)\Phi_u^*\cdot\left(\partial_\mu + i\frac{1}{\sqrt{N}}B_\mu\right)\Phi_u$$

$$+\frac{1}{g_{d,0}^2}\left(\partial_\mu - i\frac{1}{\sqrt{N}}B_\mu\right)\Phi_d^*\cdot\left(\partial_\mu + i\frac{1}{\sqrt{N}}B_\mu\right)\Phi_d \qquad (34)$$

$$+\frac{B_\mu^2}{N(4\pi)^2}\left(a_u\Phi_u^*\Phi_u + a_d\Phi_d^*\Phi_d\right) + \frac{1}{N}\frac{\lambda_{u,0}}{4g_{u,0}^4}\left(\Phi_u^*\Phi_u\right)^2 + \frac{1}{N}\frac{\lambda_{d,0}}{4g_{d,0}^4}\left(\Phi_d^*\Phi_d\right)^2$$

where

$$B_\mu \equiv -\frac{(4\pi)^2}{2\Lambda^2}\frac{1}{\sqrt{N}}\left(\overline{u}_R^I\gamma_\mu u_R^I - \overline{d}_R^I\gamma_\mu d_R^I\right) \qquad (35)$$

$$\Phi_u \equiv -\frac{(4\pi)^2}{2\Lambda^2}\frac{1}{\sqrt{N}}i\overline{u}_L^I u_R^I, \quad \Phi_d \equiv -\frac{(4\pi)^2}{2\Lambda^2}\frac{1}{\sqrt{N}}i\overline{d}_L^I d_R^I \qquad (36)$$

If we had considered a non-vanishing $\tilde{\lambda}$ in the previous section, we would have obtained a term proportional to $|\Phi_u|^2|\Phi_d|^2$ which is allowed by the global symmetry of the lagrangian.

The counterterms $\Delta\mathcal{L}$ are essential for the complete equivalence to the original chiral QED. Without $\Delta\mathcal{L}$, the matter-only lagrangian would depend only on the cutoff Λ and the three parameters v^2, V_u^2, V_d^2, and in no way the theory would be equivalent to the chiral QED which has more number of marginal parameters.

5   Chiral anomaly

The chiral QED has two massless scalar modes φ_u, φ_d, and one is used for the Higgs mechanism to give the photon a mass, but the other is left. The remaining massless mode couples to two photons, and the renormalized vertex is given by (using the results in Appendix B)

$$\left\langle \left(\frac{g_d}{m_d} \varphi_u + \frac{g_u}{m_u} \varphi_d \right) (-(k+l)) A_\alpha(k) A_\beta(l) \right\rangle$$
$$\simeq \frac{1}{\sqrt{2N}} \frac{i}{(4\pi)^2} \frac{8}{3} \frac{g_u g_d}{m_u m_d} \epsilon_{\alpha\beta\mu\nu} k_\mu l_\nu \tag{37}$$

Because of the equivalence between the chiral QED and the matter-only theory of the previous section, the same result is obtained for the correlation of the corresponding interpolating fields. In other words chiral anomaly is correctly reproduced by the matter-only theory.

6   Conclusion

The essence of the present and previous papers is that any field theory can be renormalized by fine tuning the relevant parameters. This is what the Wilsonian renormalization group tells us. Hence, given an arbitrary lagrangian, whether it is manifestly renormalizable or not, if we can find a critical point, we get a renormalizable theory by finely adjusting the relevant parameters so that the physical mass is much smaller than the cutoff. The resulting theory depends only on the relevant and marginal degrees of freedom. We have elaborated on this expectation using concrete examples of abelian gauge theories in this and previous papers. We have given a very simple procedure for constructing a matter-only lagrangian which is equivalent to the original manifestly renormalizable gauge theory. We have used the $\frac{1}{N}$ expansions to locate non-trivial critical points which would be missed if we used perturbation expansions in non-renormalizable interactions.

There has been some hope that the theories without elementary gauge fields or scalar fields would be more tightly constrained.[10] From the Wilsonian RG viewpoint, this hope is not well founded.[6] Unless the symmetry of the theory is enhanced by imposing the vanishing of a relevant or marginally irrelevant parameter, there is no justification for the relationship among the free parameters of the theory. Writing a matter-only theory in a particular form has, at best, as much significance as imposing an arbitrary relation among the parameters of the theory.

The author is aware that much of what is written in this paper may sound obvious to those versed in the Wilsonian RG. Since there is no reference to point to, however, the author hopes that this paper has its merit in describing a proper and modern way of looking at this old subject.

A   The Yukawa model vs. the NJL model

The perturbatively renormalizable Yukawa model, which is invariant under a chiral $U(1)$, is defined by

$$
\begin{aligned}
\mathcal{L}_Y &= \partial_\mu \phi^* \partial_\mu \phi + \frac{\lambda_0}{4N} (\phi^* \phi)^2 + m_0^2 \phi^* \phi \\
&\quad + \overline{\psi}^I \frac{1}{i} \partial\!\!\!/ \psi^I + i \frac{g_0}{\sqrt{N}} \left(\phi \, \overline{\psi}_R^I \psi_L^I + \phi^* \, \overline{\psi}_L^I \psi_R^I \right)
\end{aligned}
\tag{38}
$$

with a momentum cutoff Λ. By rescaling ϕ by $\frac{1}{g_0}\phi$, we rewrite the above as

$$
\begin{aligned}
\mathcal{L}_Y &= \frac{1}{g_0^2} \partial_\mu \phi^* \partial_\mu \phi + \frac{\lambda_0}{4g_0^4 N} (\phi^* \phi)^2 + \frac{m_0^2}{g_0^2} \phi^* \phi \\
&\quad + \overline{\psi}^I \frac{1}{i} \partial\!\!\!/ \psi^I + i \frac{1}{\sqrt{N}} \left(\phi \, \overline{\psi}_R^I \psi_L^I + \phi^* \, \overline{\psi}_L^I \psi_R^I \right)
\end{aligned}
\tag{39}
$$

To leading order in $\frac{1}{N}$, the renormalized parameters are given by

$$
\frac{1}{g^2} \equiv \frac{1}{(4\pi)^2} \ln \frac{\Lambda_0^2}{\mu^2} = \frac{1}{g_0^2} + \frac{1}{(4\pi)^2} \ln \frac{\Lambda^2}{\mu^2}
\tag{40}
$$

$$
\frac{m^2}{g^2} = \frac{m_0^2}{g_0^2} - \frac{2}{(4\pi)^2} \Lambda^2
\tag{41}
$$

$$
\frac{\lambda}{4g^4} = \frac{\lambda_0}{4g_0^4} + \frac{1}{(4\pi)^2} \ln \frac{\Lambda^2}{\mu^2}
\tag{42}
$$

where μ is an arbitrary low energy renormalization scale.

The theory is in the symmetric (broken) phase if

$$
\frac{m_0^2}{g_0^2} > (<) \frac{2}{(4\pi)^2} \Lambda^2
\tag{43}
$$

Following the procedure given in sect. 2, we obtain an equivalent matter-only lagrangian of the generalized Nambu-Jona-Lasinio model as

$$
\begin{aligned}
\mathcal{L}_{NJL} &= \overline{\psi}^I \frac{1}{i} \partial\!\!\!/ \psi^I + \frac{1}{Nv^2} \left(\overline{\psi}_R^I \psi_L^I \right) \cdot \left(\overline{\psi}_L^J \psi_R^J \right) \\
&\quad + \frac{1}{g_0^2} \partial_\mu \Phi^* \partial_\mu \Phi + \frac{1}{N} \frac{\lambda_0}{4 g_0^4} (\Phi^* \Phi)^2 ,
\end{aligned}
\tag{44}
$$

where $v^2 \equiv \frac{m_0^2}{g_0^2} = \frac{2}{(4\pi)^2} \Lambda^2 + \frac{m^2}{g^2}$, and

$$
\Phi \equiv -\frac{i}{\sqrt{N}} \frac{(4\pi)^2}{2\Lambda^2} \overline{\psi}_L^I \psi_R^I, \quad \Phi^* \equiv -\frac{i}{\sqrt{N}} \frac{(4\pi)^2}{2\Lambda^2} \overline{\psi}_R^I \psi_L^I
\tag{45}
$$

If we raise the cutoff Λ to the Landau scale Λ_0, we get $\frac{1}{g_0^2} \to 0$, and $\frac{\lambda_0}{4 g_0^4} \to \frac{\lambda}{4 g^4} - \frac{1}{g^2}$. Hence, we obtain a simpler lagrangian

$$
\mathcal{L}'_{NJL} = \overline{\psi}^I \frac{1}{i} \partial\!\!\!/ \psi^I + \frac{1}{Nv^2} \left(\overline{\psi}_R^I \psi_L^I \right) \cdot \left(\overline{\psi}_L^J \psi_R^J \right) + \frac{1}{N} \left(\frac{\lambda}{4 g^4} - \frac{1}{g^2} \right) (\Phi^* \Phi)^2
\tag{46}
$$

For an original discussion, see ref. [6].

B The renormalized two-, three-, and four-point functions in chiral QED

For definiteness we will summarize the results of the lowest order calculations in $\frac{1}{N}$, ignoring the contributions suppressed by negative powers of the cutoff. We only list those necessary for verifying the Ward identities.

B.1 two-point functions

For small external momenta, we obtain the following approximate results for the proper two-point functions:

$$
\begin{aligned}
\Pi_{A_\mu A_\nu}(k) &\simeq \left[\frac{1}{e^2} + \frac{1}{(4\pi)^2} \left(\frac{2}{3} \ln \frac{\mu^2}{m_u^2} + \frac{2}{3} \ln \frac{\mu^2}{m_d^2} - \frac{5}{3} \right) \right] \left(k^2 \delta_{\mu\nu} - k_\mu k_\nu \right) \\
&\quad + \frac{1}{e^2 \xi} k_\mu k_\nu + \delta_{\mu\nu} \, 2 \left[\frac{m_u^2}{g_u^2} + \frac{m_d^2}{g_d^2} \right. \\
&\quad \left. + \frac{1}{(4\pi)^2} \left\{ m_u^2 \left(\ln \frac{\mu^2}{m_u^2} - 1 \right) + m_d^2 \left(\ln \frac{\mu^2}{m_d^2} - 1 \right) \right\} \right]
\end{aligned}
\tag{47}
$$

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where $\frac{1}{\xi e^2} = \frac{1}{\xi_0 e_0^2} - \frac{1}{(4\pi)^2}\frac{1}{3}$.

$$\Pi_{\varphi_u\varphi_u}(k) \simeq k^2\left[1 + \frac{g_u^2}{(4\pi)^2}\left(\ln\frac{\mu^2}{m_u^2} - 1\right)\right] \tag{48}$$

$$\Pi_{\varphi_u A_\nu}(k) \simeq ik_\nu\sqrt{2}\frac{m_u}{g_u}\left[1 + \frac{g_u^2}{(4\pi)^2}\left(\ln\frac{\mu^2}{m_u^2} - 1\right)\right] \tag{49}$$

$$\Pi_{\rho_u\rho_u}(p) \simeq p^2\left[1 + \frac{g_u^2}{(4\pi)^2}\left(\ln\frac{\mu^2}{m_u^2} - \frac{5}{3}\right)\right]$$
$$+ \lambda_u\frac{m_u^2}{g_u^2} + 4m_u^2\frac{g_u^2}{(4\pi)^2}\left(\ln\frac{\mu^2}{m_u^2} - 1\right) \tag{50}$$

B.2 three-point functions

For small external momenta, we find

$$\Pi_{\rho_u A_\mu \varphi_u}(p, k, -(p+k)) \simeq \frac{i}{\sqrt{N}}\bigg[(2p+k)_\mu$$
$$+ \frac{g_u^2}{(4\pi)^2}\left\{k_\mu\left(\ln\frac{\mu^2}{m_u^2} - 3\right) + p_\mu\left(2\ln\frac{\mu^2}{m_u^2} - 4\right)\right\}\bigg] \tag{51}$$

$$\Pi_{\rho_u\varphi_u\varphi_u}(p, k, -(p+k)) \simeq -\frac{1}{\sqrt{2N}}\frac{g_u}{m_u}\bigg[\lambda_u\frac{m_u^2}{g_u^2}$$
$$+ \frac{g_u^2}{(4\pi)^2}\left\{4m_u^2\left(\ln\frac{\mu^2}{m_u^2} - 1\right) - 2\left(k^2 + kp + \frac{p^2}{3}\right)\right\}\bigg] \tag{52}$$

$$\Pi_{A_\mu A_\nu \rho_u}(k, l, -(k+l))$$
$$\simeq -\frac{2\sqrt{2}}{\sqrt{N}}\frac{m_u}{g_u}\delta_{\mu\nu}\left[1 + \frac{g_u^2}{(4\pi)^2}\left(\ln\frac{\mu^2}{m_u^2} - 2\right)\right] \tag{53}$$

$$\Pi_{A_\mu A_\nu \varphi_u}(k, l, -(k+l)) \simeq \frac{1}{\sqrt{2N}}\frac{i}{(4\pi)^2}\frac{4}{3}\frac{g_u}{m_u}\epsilon_{\alpha\beta\mu\nu}k_\mu l_\nu \tag{54}$$

Note Π_{AAA} vanishes due to the CP invariance.

B.3 four-point functions

For small external momenta we find

$$\Pi_{A_\alpha A_\beta A_\gamma A_\delta} \simeq \frac{1}{N}\frac{1}{(4\pi)^2}\frac{4}{3}(\delta_{\alpha\beta}\delta_{\gamma\delta} + \text{permutations}) \tag{55}$$

$$\Pi_{\varphi_u A_\alpha A_\beta A_\gamma}(k, ...) \simeq \frac{i}{N}\frac{g_u}{m_u}\frac{1}{(4\pi)^2}\frac{\sqrt{2}}{3}(\delta_{\alpha\beta}k_\gamma + \delta_{\alpha\gamma}k_\beta + \delta_{\beta\gamma}k_\alpha) \tag{56}$$

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C   Ward identities

The following Ward identities must be satisfied.

$$-ik_\mu \Pi_{A_\mu A_\nu}(k) + \sqrt{2}\frac{m_u}{g_u}\Pi_{\varphi_u A_\nu}(k) - \sqrt{2}\frac{m_d}{g_d}\Pi_{\varphi_d A_\nu}(k) = 0 \tag{57}$$

$$-ik_\mu \Pi_{A_\mu \varphi_u}(k) + \sqrt{2}\frac{m_u}{g_u}\Pi_{\varphi_u \varphi_u}(k) - \sqrt{2}\frac{m_d}{g_d}\Pi_{\varphi_d \varphi_d}(k) = 0 \tag{58}$$

$$-ik_\mu \Pi_{A_\mu \rho_u \varphi_u}(k,p,-(k+p)) + \sqrt{2}\frac{m_u}{g_u}\Pi_{\varphi_u \rho_u \varphi_u}(k,p,-(k+p))$$

$$= -\frac{1}{\sqrt{N}}\Pi_{\varphi_u \varphi_u}(p+k) + \frac{1}{\sqrt{N}}\Pi_{\rho_u \rho_u}(p) \tag{59}$$

$$-ik_\mu \Pi_{A_\mu \rho_u A_\nu}(k,p,-(p+k)) + \sqrt{2}\frac{m_u}{g_u}\Pi_{\varphi_u \rho_u A_\nu}(k,p,-(p+k))$$

$$= -\frac{1}{\sqrt{N}}\Pi_{\phi_u A_\nu}(k) \tag{60}$$

$$-i(k_\mu + l_\mu)\Pi_{A_\mu A_\alpha A_\beta}(-(k+l),k,l) \tag{61}$$

$$= -\sqrt{2}\frac{m_u}{g_u}\Pi_{\varphi_u A_\alpha A_\beta}(-(k+l),k,l) + \sqrt{2}\frac{m_d}{g_d}\Pi_{\varphi_d A_\alpha A_\beta}(-(k+l),k,l)$$

$$-ik_\alpha \Pi_{A_\alpha A_\beta A_\gamma A_\delta}(k,...)$$

$$= -\sqrt{2}\frac{m_u}{g_u}\Pi_{\varphi_u A_\beta A_\gamma A_\delta}(k,...) + \sqrt{2}\frac{m_d}{g_d}\Pi_{\varphi_d A_\beta A_\gamma A_\delta}(k,...) \tag{62}$$

References

[1] For example, W. Heisenberg, Rev. Mod. Phys. **29**(1957)269

[2] Y. Nambu, G. Jona-Lasinio, Phys. Rev. **122**(1961)345

[3] J. D. Bjorken, Ann. Phys. **24**(1963)174

[4] I. Bialynicki-Birula, Phys. Rev. **130**(1963)465;
     D. Lurié, A. J. Macfarlane, Phys. Rev. **136**(1964)B816;
     G. S. Guralnik, Phys. Rev. **136**(1964)B1404

[5] T. Eguchi, Phys. Rev. D**14**(1976)2755; Phys. Rev. D**17**(1978)611

[6] A. Hasenfratz, P. Hasenfratz, K. Jansen, J. Kuti, Y. Shen, Nucl. Phys.
     B365(1991)79

[7] K. G. Wilson, J. Kogut, Phys. Repts. **12**(1974)75 — 200

[8] H. Sonoda, "QED out of matter," hep-th/0002203

[9] For a pedagogical review on 1/N expansions, see S. Coleman, *Aspects of Symmetry* (Cambridge Univ. Press, 1985)

[10] K. Akama, Phys. Rev. Lett. **76**(1996)184, and references therein